Journal of Statistical Physics, Vol. 100, Nos. 3/4, 2000

Bounds on the Growth of the Velocity Support for the Solutions of the Vlasov–Poisson Equation in a Torus

Emanuele Caglioti[1] **and Carlo Marchioro**[1]

Received June 9, 1999; final March 27, 2000

A bound on the growth of the velocity for the Vlasov–Poisson equation in a torus is given in one and two dimensions. The main tool used in the proof is a partition into fast and slow particles and the ergodic property of the free motion in a torus.

KEY WORDS: Vlasov–Poisson equation; asymptotic behavior of the solutions.

1. INTRODUCTION

In the present paper we study a particular feature of the solution $f(\underline{x}, \underline{v}, t)$ of the Vlasov–Poisson equation in a flat torus $T_d = [0, 2\pi]^d$, where d is the dimension of the space. The Vlasov–Poisson equation reads:

$$\partial_t f + \underline{v} \cdot \nabla_x f + \underline{E}(\underline{x}, t) \cdot \nabla_v f = 0, \qquad \underline{x} \in T_d, \quad \underline{v} \in \mathbb{R}^d \tag{1.1}$$

where

$$\underline{E} = \nabla U, \qquad \Delta U = \rho - \rho^*, \tag{1.2}$$

the density of the electrons ρ is

$$\rho(\underline{x}, t) = \int_{\mathbb{R}^d} d\underline{v}\, f(\underline{x}, \underline{v}, t) \tag{1.3}$$

$$f(\underline{x}, \underline{v}, t) \geqslant 0, \qquad \int_{T_d} \int_{\mathbb{R}^d} d\underline{x}\, d\underline{v}\, f(\underline{x}, \underline{v}, t) = M > \infty, \tag{1.4}$$

and $\rho^* = (2\pi)^{-d} M$ is the constant density of the ions.

[1] Dipartimento di Matematica, Università di Roma La Sapienza, 00185 Rome, Italy; e-mail: caglioti@axcasp.caspur.it; marchioro@axcasp.caspur.it.

0022-4715/00/0800-0659$18.00/0

In the present paper we want to give a nontrivial bound on the growth of the velocity

$$V(t) = \sup |\underline{v}| \tag{1.5}$$

where the supremum is taken on the $(\underline{x}, \underline{v}) \in$ *support of* $f(\underline{x}, \underline{v}, t)$.

We suppose that initially

$$f(\underline{x}, \underline{v}, 0) = f_o(\underline{x}, \underline{v}) = 0 \qquad \text{if} \quad |\underline{v}| > V^* \quad (V^* < \infty) \tag{1.6}$$

$$\|f(\underline{x}, \underline{v}, 0)\|_\infty = \sup f_o(\underline{x}, \underline{v}) < \infty \tag{1.7}$$

We prove that there exists an α such that for any $t \geqslant 0$ we have:

$$V(t) \leqslant (C_1 + C_2 t)^\alpha \tag{1.8}$$

where from now on we denote by C_i or C any positive constant independent of time.

The goodness of the result depends on the value of α. There exist situations in which the maximal velocity remains always bounded. A trivial example is given by the stationary states. A more sophisticated example could be given by the so called "Landau damping," in which the following conjecture is made: there exist initial conditions for which the time evolution produces asymptotically a free motion. This conjecture is justified roughly by linearizing the equation and it has been rigorously proved in one dimension in ref. 3.

In general the maximal velocity could grow. How fast? For instance if we suppose $\underline{E}$ bounded (of course it is true in one dimension only, while in higher dimension there is a singularity in the interaction), then the result with $\alpha = 1$ is trivial. The goal of the present paper is to obtain $\alpha < 1$.

The interest in the present paper lies also in the technique we use. We show a non-trivial application to a system of infinitely many degrees of freedom of an idea largely used in many problems with few degrees of freedom (for instance in celestial mechanics).

More precisely the main tool in the proof is a partition into fast and slow particles. The fastest particle feels an electric field mainly produced by the slow particles. In fact, due to the fact that the kinetic energy of the system is bounded from above, the fast particles are few. The field produced by the slow particles changes on a time scale much larger than that of the fastest particle. We will show that this field, by an average effect, decreases as the maximal velocity increases.

The problem to give some bounds to the growth of the velocity support for the solutions of the Vlasov–Poisson equation has been studied as an essential tool in the proof of existence and the uniqueness theorems (for recent results in the whole space see refs. 4, 5, 9, 11, and 13, in a torus see ref. 2). Further papers study the growth of the velocity by using the dispersive properties of a plasma in $\mathbb{R}^3$ see refs. 6–8, 10. These researches study a different problem from the present one where the average effects of the motion on a torus are essential due to the absence of the dispersive property.

The solution depends on the dimension of the space. First we study in Section 2 the one-dimensional case, where the result follows from a time-average theorem only. Then in Section 3 we investigate the more interesting two-dimensional case, where we have to control the singularity of the interaction and we will use an ergodic property of the free motion. In three dimensions the singularity of the interaction is too large and we only handle with a mollified version.

2. PRELIMINARIES AND ONE-DIMENSIONAL CASE

The Vlasov–Poisson equation has been introduced in connection to the plasma physics[12] and widely studied. The characteristic curves of the Vlasov–Poisson equation are the trajectories of a charged particle interacting in the mean field of all the others. The measure $d\underline{x}\,d\underline{v}$ is conserved during the motion. Moreover

$$f(\underline{x}(t, \underline{x}_0, \underline{v}_0, t_0), \underline{v}(t, \underline{x}_0, \underline{v}_0, t_0), t) = f(\underline{x}_0, \underline{v}_0, t_0) \tag{2.1}$$

where $\underline{x}(t, \underline{x}_0, \underline{v}_0, t_0), \underline{v}(t, \underline{x}_0, \underline{v}_0, t_0)$ is the evolution at time t of the particle which at time t_0 was in $\underline{x}_0, \underline{v}_0$.

Sometimes in the sequel we shall denote $\underline{x}(t, \underline{x}_0, \underline{v}_0, t_0), \underline{v}(t, \underline{x}_0, \underline{v}_0, t_0)$ simply by $\underline{x}(t)$, $\underline{v}(t)$ when no confusion arises. In particular from (2.1) it follows:

$$\|f(\underline{x}, \underline{v}, t)\|_\infty = \|f(\underline{x}, \underline{v}, 0)\|_\infty \tag{2.2}$$

From the energy conservation and the repulsive nature of the interaction between the electrons, we have the useful inequality:

$$\int_{T_d} \int_{\mathbb{R}^d} d\underline{x}\, d\underline{v}\, |\underline{v}|^2 f(\underline{x}, \underline{v}, t) \leqslant C < \infty \tag{2.3}$$

This implies that, for any time,

$$\int_{T_d} \int_{|\underline{v}| > \bar{v}} d\underline{x}\, d\underline{v}\, f(\underline{x}, \underline{v}, t) \leqslant C\, |\bar{v}|^{-2} \tag{2.4}$$

that is, the large velocities are quite rare.

We study now the case $d=1$ and prove the following theorem:

Theorem 2.1. Consider the Vlasov–Poisson equation (1.1)–(1.4) for $d=1$ with the initial conditions (1.6), (1.7). Then the inequality (1.8) holds with $\alpha=\frac{1}{2}$.

We prove this theorem at the end of this section. The proof is an easy corollary of the following lemma:

Lemma 2.1. Consider an arbitrary time t_1 and define A_{t_1} as the set of particles (called test particles) having at this time a velocity $|\underline{v}| \geqslant V(t_1)/2$ (where $V(t)$ is defined in (1.5)). There exists a positive constant C_3, depending only on the initial data, such that if $V(t_1) \geqslant C_3$ then, for any particle in A_{t_1}

$$|v(t_1+\Delta)-v(t_1)| \leqslant C_4 \frac{\Delta}{V(t_1)} \tag{2.5}$$

where Δ is the time of the first return of the particle to the position occupied at time t_1. Moreover Δ satisfies the inequalities

$$\frac{C_5}{V(t_1)} \leqslant \Delta \leqslant \frac{C_6}{V(t_1)} \tag{2.6}$$

Proof of Lemma 2.1. Let us give some preliminary results.

First of all the electrical field is bounded by a constant C_E. Therefore, for any y, u,

$$|v(t, y, u, t_1)-u| \leqslant C_E\, |t-t_1| \tag{2.7}$$

Let us consider a test particle at $(\bar{x}, \bar{v})$. By definition $|\bar{v}| \geqslant V(t_1)/2 \geqslant C_3/2$. For the sake of simplicity let us suppose $\bar{v}>0$. In the time interval $t \in (t_1, t_1+8\pi/V)$, because of (2.7) we get

$$|v(t_1, \bar{x}, \bar{v}, t)-\bar{v}| \leqslant C_E \frac{8\pi}{V} \leqslant C_E \frac{8\pi}{C_3}$$

Taking C_3 such that $C_E < C_3^2/64\pi$ we get

$$|v(t_1, \bar{x}, \bar{v}, t)-\bar{v}| \leqslant \frac{C_3}{8} \leqslant \frac{\bar{v}}{4}$$

and hence

$$\tfrac{1}{4}\bar{v} \leqslant v(t_1, \bar{x}, \bar{v}, t) \leqslant \tfrac{5}{4}\bar{v}$$

Therefore, the lenght of the circle being 2π, and $(\bar{x}, \bar{v}) \in A_{t_1}$, we get (2.6). Substituting in (2.7) we get

$$|v(t, y, u, t_1) - u| \leqslant \frac{C}{V} \tag{2.8}$$

for any $t \in [t_1, t_1 + \Delta]$. Finally integrating with respect to time we obtain

$$|x(t, y, u, t_1) - y|_{T_1} \leqslant \frac{C}{V} \left(|u| + \frac{C}{V} \right) \tag{2.9}$$

where $|\ |_{T_1}$ denotes the distance on the circle.

We now prove Eq. (2.5). We study the growth of the velocity of a test particle by a control of the time evolution of the kinetic energy. Obviously:

$$\frac{d}{dt} \frac{|v(t)|^2}{2} = E(x, t)\, v(t) \tag{2.10}$$

So we need to control the integral of the right hand side of (2.10) along a trajectory of the particle:

$$\int_{t_1}^{t_1 + \Delta} dt\, E(x(t), t)\, v(t) \tag{2.11}$$

We remark that, the integration path being closed, this term would vanish if the electric field would be independent of time. This hypothesis, of course, is not verified, but we will show that it is "almost" verified.

We now want to evaluate (2.11).

$$\begin{aligned} &\int_{t_1}^{t_1 + \Delta} dt\, E(x(t), t)\, v(x(t), t) \\ &\quad = \int_{t_1}^{t_1 + \Delta} dt\, E(x(t), t_1)\, v(t) + \int_{t_1}^{t_1 + \Delta} dt (E(x(t), t) - E(x(t), t_1))\, v(t) \end{aligned} \tag{2.12}$$

We estimate the two terms of the right hand side of (2.12). The first term is:

$$\int_{t_1}^{t_1 + \Delta} dt\, E(x(t), t_1)\, v(t) = \int_{x(t_1)}^{x(t_1 + \Delta)} dx\, E(x, t_1) = 0 \tag{2.13}$$

In fact the integration path is closed and so the integral is equal to the difference of the potential U calculated in the same point, that is zero.

Remark the central role played by the global neutrality of the fluid. Of course this hypothesis is essential for stating the Vlasov–Poisson equation in a torus. As a consequence of the neutrality, the potential is a single-valued function.

We now evaluate the second term of the right hand side of (2.8). From (1.2) we have

$$E(x, t) = \int_0^{2\pi} dx' \, K(x - x') \, \rho(x', t) \tag{2.14}$$

where

$$K(x) = \frac{1}{2} - \frac{x}{2\pi} \tag{2.15}$$

Remark that K has a jump in $x = 0$ and is Lipschitz elsewhere.

We write explicitly the dependence on the variable at time t_1. By using (2.1) and the conservation in time of the measure, we have:

$$\begin{aligned}
&\int_{t_1}^{t_1+\Delta} dt (E(x(t), t) - E(x(t), t_1)) \, v(t) \\
&\quad = \int_{t_1}^{t_1+\Delta} dt \, v(t) \left[\int_0^{2\pi} dx' \, K(x(t) - x') \int_{\mathbb{R}} dv' \, f(x', v', t) \right. \\
&\qquad \left. - \int_0^{2\pi} dy \, K(x(t) - y) \int_{\mathbb{R}} du \, f(y, u, t_1) \right] \\
&\quad = \int_{t_1}^{t_1+\Delta} dt \, v(t) \int_0^{2\pi} dy \int_{\mathbb{R}} du \\
&\qquad \times f(y, u, t_1) [K(x(t) - x(t, y, u, t_1)) - K(x(t) - y)]
\end{aligned} \tag{2.16}$$

We make a partition of the phase space of the torus at time t_1. Of course the whole electric field is the sum of the contributions of the electric field produced by the particles in each element of the partition. We divide the particles producing the electric field E into many different classes B_h:

$$\begin{aligned}
&B_0 = \{x, v \mid |v| < a_1\} \\
&B_h = \{x, v \mid a_h \leqslant |v| < a_{h+1}\}, \quad a_{h+1} = 2a_h, \quad a_1 = v^* \quad h = 1, 2, ...
\end{aligned} \tag{2.17}$$

where v^* is a fixed quantity depending only on the initial data. In particular v^* is independent of V.

Equation (2.16) can be written as $\sum_h I_h$ where

$$I_h = \int_{t_1}^{t_1+\Delta} dt\, v(t) \int_{B_h} dy\, du\, f(y, u, t_1)[K(x(t) - x(t, y, u, t_1)) - K(x(t) - y)] \tag{2.18}$$

In each B_h we make a further division into two parts: at any time t we consider the particles y, u such that $|x(t) - y|_{T_1} \leqslant C_7 a_h V^{-1}$ and the other ones. We choose C_7 large enough in such a way that if $|x(t) - y|_{T_1} \geqslant C_7 a_h V^{-1}$ then $|x(t) - x(t, y, u, t_1)|_{T_1} \geqslant (C_7/2)\, a_h V^{-1}$. This is possible because of (2.8).

More precisely we define

$$D_{1,h} = \{(t, x, v) \colon (x, v) \in B_h,\ |x(t) - y|_{T_1} < C_7 a_h V^{-1}\} \tag{2.19}$$

and $D_{2,h}$ as the complementary set of $D_{1,h}$ in $(t_1, t_1 + \Delta) \times B_h$.

We can write

$$I_h = I_{1,h} + I_{2,h} \tag{2.20}$$

where, for $\sigma = 1, 2$

$$I_{\sigma,h} = \int_{D_{\sigma,h}} dt\, dy\, du\, f(y, u, t_1)[K(x(t) - x(t, y, u, t_1)) - K(x(t) - y)] \tag{2.21}$$

$I_{2,h}$ is easily bounded by using the Lipschitz property of K. We have

$$|I_{2,h}| \leqslant CV\Delta a_{h+1} V^{-1} \int_{B_h} dy\, du\, f(y, u, t_1) \leqslant \frac{C\Delta}{a_{h+1}} \tag{2.22}$$

where in the last inequality we have used (2.4).

We discuss now $I_{1,h}$.

$$\begin{aligned} |I_{1,h}| &\leqslant C \int_{D_{1,h}} dt\, dy\, du\, |v|\, f(y, u, t_1) \\ &\qquad \times [\,|K(x(t) - x(t, y, u, t_1))| + |K(x(t) - y)|\,] \\ &\leqslant C \int_{D_{1,h}} dt\, dy\, du\, |v|\, f(y, u, t_1) \\ &\leqslant CV \int_{D_{1,h}} f(y, u, t) \end{aligned} \tag{2.23}$$

where we used the fact that K is bounded, and (2.8).

Let us now define $m_h(t)$ as the mass of the particles in B_h such that $|x(t)-y| \leqslant C_7 a_h/V$: i.e.,

$$m_h(t) = \int_{\{(y,u) \in B_h \mid |x(t)-y| < C_7 a_h V^{-1}\}} dy\, du\, f(y,u,t_1) \tag{2.24}$$

We notice that the time in which the test particles remains in an interval of size R is bounded by $C \Delta R$. Moreover by using (2.4) we know that the total mass of the particles in B_h is bounded by C/a_h^2. Therefore we get

$$(2.21) \leqslant C\, \Delta a_h V \frac{1}{a_h^2} \tag{2.25}$$

The sum in h is bounded because a_h increase exponentially and therefore we get (2.5). ▮

Proof of Theorem 2.1. The further steps to obtain Theorem 2.1 are easy. Let us notice that (2.5) implies that

$$v^2(t+\Delta) - v^2(t) \leqslant 3C_4 \Delta \tag{2.26}$$

Let us assume (1.8) with $\alpha = 1/2$ true until time $t_1 \geqslant A > 0$. Let us notice that, the electric field being bounded this assumption can be satisfied by choosing $C_1 > \sqrt{C_3}$ large enough. Let T be the infimum of the set of times for which (1.8) is false. Then for any $t \leqslant T$ (1.8) is satisfied, while there exists a sequence $\tau_k > 0$ ($\tau_k \to 0$ *as* $k \to \infty$), such that (1.8) is false at time $T + \tau_k$. Let us consider a particle with velocity v such $|v(T+\tau_k)| > V(T+\tau_k) - \varepsilon$, $\varepsilon > 0$. We evolve it backward up to the first time, $T + \tau_k - \Delta_k$, in which it returns to the same position. Because of (2.6) Δ_k is bounded from below and therefore we get $T + \tau - \Delta_k < T$ for k large enough. If k and A are large enough and ε is small enough $v(T+\tau_k - \Delta_k)$ satisfies the hypothesis of Lemma 2.1. Then from (2.27) and (1.8) we get

$$v^2(T+\tau_k) \leqslant C_1 + C_2(T+\tau_k - \Delta_k) + 3C_4 \Delta_k + O(\varepsilon)$$

which is, choosing $C_2 > 3C_4$, in contradiction with

$$v^2(T+\tau_k) \geqslant C_1 + C_2(t+\tau_k)$$

as $k \to \infty$ and $\varepsilon \to 0$. ▮

3. TWO-DIMENSIONAL CASE

In two dimensions some new features arise: the interaction is not anymore bounded and the path of the test particle may not be closed. The electrical field $\underline{E}$ is defined by

$$\underline{E}(\underline{x}, t) = \int_{T^2} d\underline{x}' \; \underline{K}(\underline{x}, \underline{x}')(\rho(\underline{x}', t) - \rho^*) \tag{3.1}$$

Here $\underline{K}(\underline{x} - \underline{x}') = \nabla_{\underline{x}} G(\underline{x} - \underline{x}')$, where G is the Green function of the Laplace operator in T_2:

$$\underline{K}(\underline{x} - \underline{x}') = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\sum_{\underline{i}:\, \|\underline{i}\| = k} \nabla_x \ln |\underline{x} - \underline{x}' - 2\pi \underline{i}| \right), \qquad \|\underline{i}\| = \max(|i_1|, |i_2|) \tag{3.2}$$

Before proving the main result of this section let us give some useful preliminary results.

In this paper we do not use the explicit form of $\underline{K}$ but only the properties:

$$|\underline{K}(\underline{x} - \underline{x}')| \leqslant \frac{C}{|\underline{x} - \underline{x}'|_{T_2}} \tag{3.3}$$

$$|\underline{K}(\underline{x} - \underline{x}') - \underline{K}(\underline{x} - \underline{x}'')| \leqslant \frac{C\, |\underline{x}' - \underline{x}''|_{T_2}}{\min(|\underline{x} - \underline{x}'|_{T_2}^2, |\underline{x} - \underline{x}''|_{T_2}^2)} \tag{3.4}$$

where $|\cdot|_{T_2}$ denotes the distance in the flat torus.

Therefore the interaction between the electrons is not bounded. However the electric field $\underline{E}$ produced is not very large. Indeed inequality (2.3) implies that for any time

$$\int_{T_2} d\underline{x}\; |\rho(\underline{x}, t)|^2 < \infty \tag{3.5}$$

In fact for any constant P we have:

$$\begin{aligned} \rho(\underline{x}, t) &= \int_{|\underline{v}| \leqslant P} d\underline{v}\; f(\underline{x}, \underline{v}, t) + \int_{|\underline{v}| > P} d\underline{v}\; f(\underline{x}, \underline{v}, t) \\ &\leqslant \pi P^2 \, \|f(\underline{x}, \underline{v}, t)\|_\infty + P^{-2} \int_{\mathbb{R}^2} d\underline{v}\; |\underline{v}|^2 f(\underline{x}, \underline{v}, t) \end{aligned} \tag{3.6}$$

We choose P such that the right hand side of (3.6) is smallest, and by using (2.2) we have:

$$\rho(\underline{x}, t) \leqslant C \left(\int_{\mathbb{R}^2} d\underline{v}\, |\underline{v}|^2 f(\underline{x}, \underline{v}, t) \right)^{1/2} \tag{3.7}$$

Then, by using (2.3), we obtain (3.5).

The bound (3.5) allows us to control the value of $\underline{E}$ by $V(t)$ ($V(t)$ is defined in (1.5)). In fact

$$\begin{aligned} \|\underline{E}(\underline{x}, t)\|_\infty &= \sup_{\underline{x}} |\underline{E}(\underline{x}, t)| \\ &\leqslant C\, \|\rho(\underline{x}, t)\|_\infty\, R + \left(\int_{T_2} d\underline{x}\, |\rho(\underline{x}, t)|^2 \right)^{1/2} \\ &\quad \times \left(\int_{|\underline{x}'| \geqslant R} d\underline{x}'\, |\underline{x}'|^{-2} \right)^{1/2} + C \end{aligned} \tag{3.8}$$

We choose $R = (1 + \|\rho(\underline{x}, t)\|_\infty)^{-1}$ and we observe that (2.2) implies

$$\|\rho(\underline{x}, t)\|_\infty \leqslant C V(t)^2 \tag{3.9}$$

from which

$$\|\underline{E}(\underline{x}, t)\|_\infty \leqslant C + C(\log(1 + V(t)))^{1/2} \tag{3.10}$$

This estimate, which plays an important role in the sequel, tells us that the fast particles move for short time following almost the free motion.

Finally let us give the following bound on the electric field.

Lemma 3.1. Let us consider a subset Λ of $T_2 \times R^2$. Given $f(\underline{x}, \underline{v}) > 0$ let us define m_Λ, ρ_Λ, $\rho_{\Lambda, \infty}$, as

$$m_\Lambda = \int d\underline{x}\, d\underline{v}\, f(\underline{x}, \underline{v})\, \chi_\Lambda(\underline{x}, \underline{v})$$

$$\rho_\Lambda(\underline{x}) = \int d\underline{v}\, f(\underline{x}, \underline{v})\, \chi_\Lambda(\underline{x}, \underline{v})$$

$$\rho_{\Lambda, \infty} = \sup_{\underline{x}} \rho_\Lambda(\underline{x})$$

where χ_Λ is the characteristic function of the set Λ.

Then the electric field produced by the particles in Λ is bounded by

$$|\underline{E}| \leqslant C\sqrt{m_\Lambda \rho_{\Lambda,\infty}} \tag{3.11}$$

Proof. We notice that $\underline{K}$ defined in (3.2) can be written as $\underline{K}(\underline{x}-\underline{y}) = (1/2\pi)[(\underline{x}-\underline{y})/|\underline{x}-\underline{y}|^2] + \gamma(\underline{x}-\underline{y})$, where γ is smooth. Therefore we have

$$|\underline{E}(\underline{x})| \leqslant \int \left(\frac{1}{2\pi}\frac{1}{|\underline{x}-\underline{y}|} + C\right) \rho_\Lambda(\underline{y})\, d\underline{y} \leqslant \int \frac{1}{2\pi}\frac{1}{|\underline{x}-\underline{y}|}\, \rho_\Lambda(\underline{y})\, d\underline{y} + Cm_\Lambda$$

To obtain the largest value of the right handed side we rearrange the particles closest as possible around $\underline{x}$ in a circle of radius R. With this rearrangement it is easy to evaluate the intensity, which results less than $C\rho_{\Lambda,\infty} R + Cm_\Lambda$. By using the fact that m_Λ is larger than $CR^2\rho_{\Lambda,\infty}$, we get

$$|\underline{E}| \leqslant C\sqrt{m_\Lambda \rho_{\Lambda,\infty}} + Cm_\Lambda$$

Since $m_\Lambda \leqslant (2\pi)^2 \rho_{\Lambda,\infty}$, we achieve the proof. ∎

We prove the following theorem:

Theorem 3.1. Consider the Vlasov–Poisson equation (1.1)–(1.4) with the initial conditions (1.6), (1.7), then the inequality (1.8) holds for any $\alpha > \frac{6}{7}$.

As in one dimension, we start by giving a bound on the growth of the velocity in a short time Δ. Then the theorem follows as an easy consequence.

Lemma 3.2. Consider an arbitrary time t_1 and define A_{t_1} as the set of particles (called test particles) having at this time a velocity $|\underline{v}| \geqslant V(t_1)/2$.

For any $\gamma < 1/6$ there exist two positive constants C_3, C_γ, (depending only on the initial data) such that if $V(t_1) \geqslant C_3$ then, for any particle in A_{t_1}

$$\big||v(t_1+\Delta)| - |v(t_1)|\big| \leqslant C_\gamma \frac{\Delta}{V^\gamma(t_1)} \tag{3.12}$$

for some Δ which satisfies

$$\frac{C_4}{V(t_1)} \leqslant \Delta \leqslant \frac{C_5}{V(t_1)} \tag{3.13}$$

Proof of Lemma 3.2. In the proof, when no misunderstandings are possible, we denote $V(t_1)$ by V. Let us consider a particle in A_{t_1} (called in the sequel test particle) with position $\underline{x}(t_1)=\tilde{\underline{x}}$ and velocity $\underline{v}(t_1)=\tilde{\underline{v}}$. Without loss of generality we suppose $|\tilde{v}_1| \geqslant |\tilde{v}_2|$ and $\tilde{v}_1 > 0$. We study the motion of the test particle.

Let us now consider the free motion $\underline{x}_f(t)=\tilde{\underline{x}}+(t-t_1)\,\tilde{\underline{v}}$. At time $\tau_k = t_1 + (2\pi/\tilde{v}_1)\,k$, $k=1, 2, 3,...$ we get $x_{f,1}(\tau_k)=\tilde{x}_1$ (i.e., the particle has the same abscissa) and

$$x_{f,2}(\tau_k)=x_{f,2}(t_1)+\alpha k=\tilde{x}_2+\alpha k \tag{3.14}$$

where $\alpha=\tilde{v}_2/\tilde{v}_1$.

Therefore the evolution of $\underline{x}_{f,2}(\tau_k)$ is a shift on the circle of given quantity α that is a Jacobi annulus.

By Poincaré's theorem, see, e.g., ref. 1, we know that for any $\varepsilon > 0$ there exists a $k_\varepsilon \leqslant 2\pi/\varepsilon$ such that

$$|x_{f,2}(\tau_{k_\varepsilon})-\tilde{x}_2|_{T_2}\leqslant \varepsilon \tag{3.15}$$

Denote τ_{k_ε} as $t_1+\Delta$. By (3.15) we have $\Delta \leqslant 2\pi/\varepsilon\tilde{v}_1$, and by the fact that $k \geqslant 1$ we have $\Delta \geqslant 2\pi/\tilde{v}_1$. Therefore

$$C_6 V^{-1} < \Delta < C_7 V^{-1}\varepsilon^{-1} \tag{3.16}$$

which is (3.13). Now we choose $\varepsilon = V^{-1/6}$ and we consider the true trajectory. By using (3.10) we observe that, for short times, its motion differs very little from a free one. In fact, for any $(\underline{y}, \underline{u})$, and for any $t \in [0, \Delta]$,

$$|\underline{v}(t, \underline{y}, \underline{u}, t_1)-\underline{u}| \leqslant (C + C(\log V)^{1/2})\,\Delta \leqslant 1 \tag{3.17}$$

$$|x(t, \underline{y}, \underline{u}, t_1)-(\underline{y}+\underline{u}t)|_{T^2} \leqslant (C + C(\log V)^{1/2})\,\Delta^2 \leqslant \Delta \tag{3.18}$$

Therefore we have,

$$\begin{aligned} |\underline{x}(\tau_{k_\varepsilon}, \tilde{\underline{x}}, \tilde{\underline{v}}, t_1)-\tilde{\underline{x}}|_{T_2} &\leqslant |\underline{x}(\tau_{k_\varepsilon}, \tilde{\underline{x}}, \tilde{\underline{v}}, t_1)-x_f(\tau_{k_\varepsilon})|_{T_2}+|x_f(\tau_{k_\varepsilon})-\tilde{\underline{x}}|_{T_2} \\ &\leqslant \Delta+\varepsilon \leqslant 2\varepsilon \leqslant 2V^{-1/6} \end{aligned} \tag{3.19}$$

As in the one-dimensional case, we must control the kinetic energy. Hence the term

$$\begin{aligned} J &= \int_{t_1}^{t_1+\Delta} dt\, \underline{E}(\underline{x}(t), t)\cdot \underline{v}(t) \\ &= \int_{t_1}^{t_1+\Delta} dt\, \underline{E}(\underline{x}(t), t_1)\cdot \underline{v}(t) + \int_{t_1}^{t_1+\Delta} dt(\underline{E}(\underline{x}(t), t)-\underline{E}(\underline{x}(t), t_1))\cdot \underline{v}(t) \end{aligned} \tag{3.20}$$

We evaluate the first term. We observe that the integral of $\underline{E} \cdot \underline{v}\, dt$ along a closed curve vanishes, because the electric field is produced by a potential. Hence

$$\int_{t_1}^{t_1+\Delta} dt\, \underline{E}(\underline{x}(t), t_1) \cdot \underline{v}(t) = \int_{\Gamma} \underline{E} \cdot d\underline{l} \tag{3.21}$$

where Γ is a curve connecting $\underline{x}(t_1)$ with $\underline{x}(t_1+\Delta)$. We choose Γ as the segment connecting the starting point with the first return point, and hence, by (3.10) and (3.19), we have:

$$\left| \int_{t_1}^{t_1+\Delta} dt\, \underline{E}(\underline{x}(t), t_1) \cdot \underline{v}(t) \right| \leqslant C(\log V)^{1/2}\, \varepsilon \leqslant C(\log V)^{1/2}\, V^{-1/6} \tag{3.22}$$

We study the second term in the right hand side of (3.20). As in one dimension we write this term as

$$\int_{t_1}^{t_1+\Delta} dt\, \underline{v}(t) \cdot \int_{T^2} d\underline{y} \int_{\mathbb{R}^2} d\underline{u}\, f(\underline{y}, \underline{u}, t)[\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)) - \underline{K}(\underline{x}(t) - \underline{y})] \tag{3.23}$$

The strategy of the proof consists in making a partition, at time t_1, of the particles producing the electric field into different sets: the slow particles, the fast particles with a velocity far from that of the test particle, the particles with a velocity close to that of the test particle. In each situation we proceed differently as we will see. More precisely let us define

$$\begin{aligned} A_1 &= \{(y, u) \colon |u| \leqslant V^{1/2}\} \\ A_2 &= \{(y, u) \colon |u| > V^{1/2},\ |u - v(t_1)| > V^{1/2}\} \\ A_3 &= \{(y, u) \colon |u - v(t_1)| \leqslant V^{1/2}\} \end{aligned} \tag{3.24}$$

and, for $i = 1, 2, 3$,

$$J_i = \int_{t_1}^{t_1+\Delta} dt\, \underline{v}(t) \cdot \int_{A_i} d\underline{y}\, d\underline{u}\, f(\underline{y}, \underline{u}, t)[\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)) - \underline{K}(\underline{x}(t) - \underline{y})] \tag{3.25}$$

Particles in A_1. In this set it is useful to introduce a further partition of the phase space into sets B_h: a particle belongs to B_h if its velocity $\underline{v}$ at time t_1 is such that

$$\begin{aligned} B_0 &= \{x, v \mid |v| < a_1\} \\ B_h &= \{x, v \mid a_h \leqslant |v| < a_{h+1}\}, \quad a_{h+1} = 2a_h, \quad a_1 = v^* \quad h = 1, 2,... \end{aligned} \tag{3.26}$$

where $v^* > 1$ is a fixed quantity depending only on the initial data. In particular v^* is independent of V.

J_1 can be written as $\sum_{h=1}^{h^*} I_h$ where $v^* 2^{h^*} \leqslant V(t_1)^{1/2}$, and

$$I_h = \int_{t_1}^{t_1+\Delta} dt\, \underline{v}(t) \cdot \int_{B_h} d\underline{y}\, d\underline{u}\, f(\underline{y}, \underline{u}, t_1)[\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)) - \underline{K}(\underline{x}(t) - \underline{y})] \tag{3.27}$$

We evaluate the interaction produced on the test particle by the particles in B_h. First of all let us notice that if a particle $(\underline{y}, \underline{u})$ belongs to B_h then by (3.17) its velocity in the time interval $(t_1, t_1 + \Delta)$ is bounded by $3a_h$ and then it moves less then $3a_h \Delta$. In fact

$$|\underline{v}(t_1, \underline{y}, \underline{u}, t)| \leqslant 2a_h + 1 \leqslant 3a_h \tag{3.28}$$

We divide this electric field into two parts: the electric field produced by the particles closer than R and the other ones. We choose R quite small with respect to the macroscopic dimensions but large enough to be sure that a particle in A_1 during the time Δ moves less than $R/2$. Using (3.28) we can satisfy this requirement choosing

$$R = 6V^{-1/3} \tag{3.29}$$

This last assumption allows us to evaluate easily the effect of these particles by using the Lipschitz condition. For close particles we use the fact that the test particle remains in a circle of radius R for a short time only.

More precisely we define

$$D_{1,h} = \{(t, \underline{x}, \underline{v}) : (\underline{x}, \underline{v}) \in B_h, |\underline{x}(t) - \underline{y}| < R\} \tag{3.30}$$

and $D_{2,h}$ as the complementary set of $D_{1,h}$ in $(t_1, t_1 + \Delta) \times B_h$.

We can write

$$I_h = I_{1,h} + I_{2,h} \tag{3.31}$$

where, for $\sigma = 1, 2$,

$$I_{\sigma,h} = \int_{D_{\sigma,h}} dt\, d\underline{y}\, d\underline{u}\, \underline{v}(t) \cdot f(\underline{y}, \underline{u}, t_1)[K(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)) - K(\underline{x}(t) - \underline{y})] \tag{3.32}$$

$I_{2,h}$ is easily bounded by using the Lipschitz condition (3.4).

We have

$$|I_{2,h}| \leqslant C \frac{V(t_1)\, \Delta^2 m_h a_h}{R^2} \tag{3.33}$$

where m_h denotes the mass of the particles which belong to B_h.

We discuss now $I_{1,h}$. A first bound is obtained by using the modulus:

$$\begin{aligned} |I_{1,h}| \leqslant C \int_{t_1}^{t_1+\Delta} dt\, |\underline{v}(t)| \int_{B_{1,h}} d\underline{y}\, d\underline{u}\, f(\underline{y}, \underline{u}, t_1) \\ \times [\,|\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1))| + |\underline{K}(\underline{x}(t) - \underline{y})|\,] \end{aligned} \tag{3.34}$$

We observe that, by virtue of (3.28)–(3.29), $|x(t) - y|_{T_2} < R$ implies $|\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)|_{T_2} < 3/2\, R$. Hence we have

$$|I_{1,h}| \leqslant G_{1,h} + G_{2,h}$$

where

$$\begin{aligned} G_{1,h} &= CV \int_{D_{1,h}} dt\, d\underline{y}\, d\underline{u}\, f(\underline{y}, \underline{u}, t_1) |\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1))| \\ G_{2,h} &= CV \int_{D_{1,h}} dt\, d\underline{y}\, d\underline{u}\, f(\underline{y}, \underline{u}, t_1) |\underline{K}(\underline{x}(t) - \underline{y})| \end{aligned} \tag{3.35}$$

Let us evaluate the first term. By using the Lemma 3.1 we have

$$G_{1,h} \leqslant CV \int_{t_1}^{t_1+\Delta} dt (m_h(t)\, \rho_{h,\infty}(t))^{1/2} \tag{3.36}$$

where $m_h(t)$ denotes the mass of the particle of B_h at distance less than $3/2\, R$ from the test particle, and $\rho_{h,\infty}(t)$ denotes the sup of the density of the particles in B_h. By using the Cauchy–Schwarz inequality

$$G_{1,h} \leqslant CV\Delta^{1/2} \left(\int_{t_1}^{t_1+\Delta} dt\, m_h(t)\, \rho_{h,\infty}(t) \right)^{1/2} \tag{3.37}$$

We have arrived at a central point in the proof. The time for which the test particle remains in a circle of radius CR is proportional to ΔR and so, by using (2.4) and the fact that $\rho_{h,\infty} \leqslant C(a_h)^2$, we have:

$$G_{1,h} \leqslant CV\Delta a_h (m_h(t)\, R)^{1/2} \leqslant CV\Delta R^{1/2} \tag{3.38}$$

We evaluate in the same way $G_{2,h}$, we add (3.33) and we obtain:

$$|I_h| \leqslant CV\Delta \left[\frac{\Delta m_h a_h}{R^2} + R^{1/2} \right] \tag{3.39}$$

Summing on h we get

$$|J_1| \leqslant \sum_{h=0}^{h^*} |I_h| \leqslant CV\Delta(V^{-1/6} \log V) \tag{3.40}$$

where we have used the fact that h^* is bounded by

$$2^{h^*} \leqslant CV^{1/2} \tag{3.41}$$

Particles in A_2. In this case the Lipschitz condition (3.4) is useless. We write

$$|J_2| \leqslant L_1 + L_2$$

where

$$\begin{aligned} L_1 &= \int_{t_1}^{t_1+\Delta} dt \int_{A_2} d\underline{y}\, d\underline{u}\, |\underline{v}(t)|\, f(\underline{y}, \underline{u}, t_1)\, |\underline{K}(\underline{x}(t) - \underline{x}(t, \underline{y}, \underline{u}, t_1)| \\ L_2 &= \int_{t_1}^{t_1+\Delta} dt \int_{A_2} d\underline{y}\, d\underline{u}\, |\underline{v}(t)|\, f(\underline{y}, \underline{u}, t_1)\, |\underline{K}(\underline{x}(t) - \underline{y})| \end{aligned} \tag{3.42}$$

Let us evaluate L_1. We follow the test particle, and at any time we consider many circles with center in the test particle and radius $r_i = V^{-2} 2^{i-1}$, $i = 0, 1,..., i^*$. We calculate the electric field produced by the particles in the first circle by Lemma 3.1 and the electric fields produced by the particles in the annuli by the bound (3.3). We have:

$$L_1 \leqslant CV \sum_{i=0}^{i^*} \int_{t_1}^{t_1+\Delta} dt\, m_{2,i}(t)[r_i]^{-1} + CV \int_{t_1}^{t_1+\Delta} dt\, (m_{2,0}(t)\, \rho_{2,\infty}(t))^{1/2} \tag{3.43}$$

where $m_{2,i}(t)$ denotes the mass of the particles of A_2 in the annulus with center in the test particle and radii r_i, r_{i+1}; $m_{2,0}$ denotes the mass of the particle of A_2 contained in a circle with center in the test particle and radius r_0; $\rho_{2,\infty}$ denotes the sup in $\underline{x}$ and in $t \in [t_1, t_1 + \Delta]$ of the density of the particles in A_2. The sum covers the whole torus: $2^{i^*} = CV^2$. Using the Cauchy–Schwarz inequality in the second term we get

$$L_1 \leqslant CV \sum_{i=0}^{i*} \int_{t_1}^{t_1+\Delta} dt\, m_{2,i}(t)[r_i]^{-1}$$
$$+ CV(t_1)\, \Delta^{1/2} \left(\int_{t_1}^{t_1+\Delta} dt\, m_{2,0}(t)\, \rho_{2,\infty}(t) \right)^{1/2} \tag{3.44}$$

We will use here the important fact that a particle of A_2 remains in a circle of radius r_i for a time τ_i which is bounded by $C\Delta V^{1/2} r_i$. Indeed the test particle moves for a distance bounded by $CV\Delta$. We divide this distance in pieces of length 2π and we have, the relative velocity of the two particles being larger than $CV^{1/2}$ in any piece, that the particle in A_2 cannot stay in the circle of radius r_i longer than $(C/V^{1/2})\, r_i$. The number of pieces is smaller than $CV\Delta$ and therefore we get the bound. More precisely, let us suppose, for the sake of simplicity, that $|v_1(t_1) - u_1| \geqslant |v_2(t_1) - u_2|$ and $v_1(t_1) - u_1 > 0$. From (3.24) we get

$$\frac{1}{\sqrt{2}}\, V(t_1)^{1/2} \leqslant v_1(t_1) - u_1 \leqslant 2V(t_1) \tag{3.45}$$

Moreover by (3.17)

$$\tfrac{1}{2} V(t_1)^{1/2} \leqslant v_1(t_1) - v_1(t, y, u, t_1) \leqslant 4V(t_1) \tag{3.46}$$

for any $t \in [t_1, t_1 + \Delta]$.

Let us define

$$z(t) = x_1(t_1) - y_1 + \int_{t_1}^{t} ds (v_1(s) - v_1(s, y, u, t_1)) \tag{3.47}$$

The two particles can interact only if $|z(t)_{\text{mod } 2\pi}| \leqslant r_i$, i.e., if $\min_{k \in \mathbb{Z}} |z(t) - 2k\pi| \leqslant r_i$.

Then τ_i is bounded by

$$\sum_{k \in \mathbb{Z}} \int_{t_1}^{t_1+\Delta} dt\, \chi(|z(t) - 2k\pi| \leqslant r_i)$$
$$\leqslant \frac{2}{V^{1/2}} \sum_{k \in \mathbb{Z}} \int_{z(t_1)}^{z(t_1+\Delta)} dz\, \chi(|z(t) - 2k\pi| \leqslant r_i)$$
$$\leqslant \frac{2}{V^{1/2}} \left(\frac{z(t_1+\Delta) - z(t_1)}{2\pi} + 1 \right) 2r_i$$
$$\leqslant \frac{2}{V^{1/2}} \left(\frac{4V\Delta}{2\pi} + 1 \right) 2r_i \leqslant CV^{1/2} \Delta r_i \tag{3.48}$$

where $\chi(A)$ denotes the characteristic function of the set A.

We return now to (3.44). Using the previous estimate we obtain:

$$L_1 \leqslant CV\Delta i^* m_2 \Delta V^{1/2} + CV\Delta^{1/2}(m_2 \rho_{2,\infty} r_0 \Delta V^{1/2})^{1/2}$$
$$\leqslant CV^{1/2} \log V\Delta + CV^{3/4}\Delta \leqslant CV^{3/4}\Delta \tag{3.49}$$

where m_2 denotes the mass in A_2, and where we have used the facts that $m_2 \leqslant C/V$ and $\rho_{2,\infty} \leqslant CV^2$. We evaluate in the same way L_2 and we get

$$|J_2| \leqslant CV^{3/4}\Delta \tag{3.50}$$

Particles in A_3. In this case the density is bounded by $CV(t_1)$ and the mass is very small. More precisely denoting with m_3 the mass of the particles in A_3 we get, by (2.4), $m_3 \leqslant C/V^2$. By using Lemma 3.1 we easily obtain

$$|J_3|\ CV(\rho_{3,\infty} m_3)^{1/2}\ \Delta \leqslant CV(VV^{-2})^{1/2}\ \Delta = CV^{1/2}\Delta \tag{3.51}$$

where $\rho_{3,\infty}$ denotes the sup in $\underline{x}$ and in $t \in [t_1, t_1 + \Delta]$ of the density of the particles in A_3.

We add J_1, J_2, J_3 and (3.22) getting

$$|J| \leqslant C\Delta V^{5/6} \log V \tag{3.52}$$

Taking into account that J is the variation of $v^2/2$ in the time interval $[t_1, t_1 + \Delta]$ we get the inequality (3.12). ▌

Proof of Theorem 3.1. We can notice that (3.12) implies that for any $\eta > 7/6$ there exists C_η such that

$$\big||V(t_1 + \Delta)|^\eta - |V(t_1)|^\eta\big| \leqslant C_\eta \Delta \tag{3.53}$$

Then we proceed exactly as in the one dimensional case. ▌

Finally a short remark on the three-dimensional case. We are not able to repeat the previous proof in this situation because in this case the interaction is too singular. In fact we can solve the equation (1.2) and, as well known, we obtain an interaction in which $|\underline{K}(\underline{x} - \underline{x}')|$ behaves as $|\underline{x} - \underline{x}'|_{T_3}^{-2}$, when $|\underline{x} - \underline{x}'|_{T_3} \to 0$ ($|\ |_{T_3}$ denotes the distance on the three dimensional torus). So to prove a Theorem with $\alpha < 1$, we can mollify the interaction by introducing a cutoff which bounds the interaction at small distances. The other steps in the proofs are similar to that of the two-dimensional case.

ACKNOWLEDGMENTS

This work was performed under the auspices of the Italian Council of Researches (C.N.R.–G.N.F.M.) and the Ministry of the University (M.U.R.S.T.).

REFERENCES

1. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, Vol. 60 (Springer-Verlag, 1997).
2. J. Batt and G. Rein, Global classical solutions of the periodic Vlasov–Poisson system in three dimension, *C. R. Acad. Sci. Paris Serie I* **313**:411–416 (1991).
3. E. Caglioti and C. Maffei, Time asymptotics for solutions of Vlasov–Poisson equation in a circle, *J. Statist. Phys.* **92**:301–321 (1998).
4. E. Horst, On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation, I. General theory, *Math. Meth. Appl. Sci.* **3**:229–248 (1981).
5. E. Horst, On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation, II. Special cases, *Math. Meth. Appl. Sci.* **4**:19–32 (1982).
6. E. Horst, On the asymptotic growth of the solutions of the Vlasov–Poisson system, *Math. Meth. Appl. Sci.* **16**:75–85 (1993).
7. R. Illner and G. Rein, Time decay of the solutions of the Vlasov–Poisson system in the plasma physical case, *Math. Meth. Appl. Sci.* **19**:1409–1413 (1996).
8. P. L. Lions and B. Perthame, Propagation of the moments and regularity for the 3-dimensional Vlasov–Poisson system, *Invent. Math.* **105**:415–430 (1991).
9. K. Pfaffelmoser, Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *J. Differ. Eq.* **95**:281–303 (1992).
10. G. Rein, Growth Estimates for the solutions of the Vlasov–Poisson system in the plasma physics case, *Mathematishe Nachrichten* **191**:269–278 (1998).
11. J. Schaeffer, Global existence of smooth solutions to the Vlasov poisson system in three dimension, *Comm. Part. Diff. Equat.* **16**:1313–1335 (1991).
12. A. Vlasov, On the kinetic theory of an assembly of particle with collective interaction, *Acad. Sci. USSR J. Phys.* **9**:25 (1945).
13. S. Wollman, Global-in-time solutions to the three-dimensional Vlasov–Poisson system, *J. Math. Anal. Appl.* **176**:76–91 (1993).